# Periodic Homogenization for Hypoelliptic Diffusions

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**Abstract**

We study the long time behavior of an Ornstein–Uhlenbeck process under the influence of a periodic drift. We prove that, under the standard diffusive rescaling, the law of the particle position converges weakly to the law of a Brownian motion whose covariance can be expressed in terms of the solution of a Poisson equation. We also derive upper bounds on the convergence rate.

## 1 Introduction

In this paper we study the long time behavior of solutions of the following Langevin equation:

$$\tau\ddot{x}(t) = v(x(t)) - \dot{x}(t) + \sigma\dot{\beta}(t)\;, \quad x(t) \in \mathbb{R}^n\;, \tag{1.1}$$

where $\beta(t)$ is a standard Brownian motion and $\sigma$, $\tau > 0$. The parameter $\tau$ can be thought of as a nondimensional particle relaxation time, which measures the inertia of the particle. The drift term $v$ is taken to be smooth, periodic with period $1$ in all directions; further, it is assumed that it satisfies an appropriate centering condition.

It is well known that as $\tau$ tends to $0$ the solution of (1.1) converges with probability $1$ to the solution of the Smoluchowski equation

$$\dot{z}(t) = v(z(t)) + \sigma\dot{\beta}(t)\;, \quad x(t) \in \mathbb{R}^n\;, \tag{1.2}$$

uniformly over every finite time interval, see e.g. [Nel67, Ch 10]. The problem of homogenization for equation (1.2) has been studied extensively over the last three decades for periodic [BLP78, Bat85, Par99] as well as random [CX97, KO01, LOY98] drifts. For the case where $v(z)$ is a smooth, periodic field which is centered with respect to the invariant measure of the process, it is not hard to prove [BLP78, Ch 3] that the rescaled process $\varepsilon z(t/\varepsilon^2)$ converges, as $\varepsilon$ tends to $0$, to a Brownian motion with a positive definite covariance matrix $\mathcal{K}$. The proof of this functional central limit theorem is based on estimates on the spectral gap of the generator of the process $z(t)$.

The long time behavior of particles with non–negligible inertia, whose evolution is governed by equation (1.1) has been investigated by Freidlin and coworkers in a series

of papers [FW98, Fre01, FW01, FW99]. Among other things, Hamiltonian systems under weak deterministic and random perturbations were studied in these papers:

$$\tau\ddot{x} = -\nabla V(x) + \varepsilon(-\kappa\dot{x} + \gamma) + \sqrt{\varepsilon}\sigma\dot{\beta}, \tag{1.3}$$

with $\kappa, \gamma \in \mathbb{R}$. It was shown that, under appropriate assumptions on the potential $V(x)$, the rescaled process $\{x(t/\varepsilon), y(t/\varepsilon)\}$ converges weakly to a diffusion process on a graph corresponding to the Hamiltonian of the system $H = \frac{1}{2}\tau\dot{x}^2 + V(x)$.

On the other hand, the problem of homogenization for (1.1) has been investigated less. This is not surprising since the hypoellipticity of the generator of the process (1.1) renders the derivation of the necessary spectral gap estimates more difficult. Homogenization results for the solution $x(t)$ of (1.1) have been obtained, to our knowledge, only for the case where the drift $v(x)$ is the gradient of a potential. In this case the invariant measure of the process $\{x(t), \dot{x}(t)\}$ is explicitly known and this fact simplifies considerably the analysis. This problem was analyzed for periodic [Rod89] as well as random potentials [PV85]. In both cases it was shown that the particle position converges, under the diffusive rescaling, to a Brownian motion with a positive covariance matrix $\mathcal{K}$. The proofs of these homogenization theorems are based on the techniques developed for the study of central limit theorems for additive functionals of Markov processes [KV86], together with a regularization procedure for appopriate degenerate Poisson equations. Related questions for subelliptic diffusions have also been investigated [Nor94, Nor97, BBJR95].

The purpose of this paper is to prove a central limit theorem for the solution of the Langevin equation (1.1) with a general periodic smooth drift $v(x)$ and, further, to obtain bounds on the convergence rate. The proof of our homogenization theorem relies on the strong ergodic properties of hypoelliptic diffusions. The techniques developed in [EPRB99, EH00] enable us to prove the existence of a unique, smooth invariant measure for (1.1) and to obtain precise estimates on the solution of the Poisson equation $-\mathcal{L}f = g$, where $\mathcal{L}$ is the generator of the process (1.1) and the function $g$ is smooth and centered with respect to the invariant measure. Based on these estimates it is rather straightforward to show that the rescaled particle position $\varepsilon x(t/\varepsilon^2)$ convergences to a Brownian motion, using the techniques developed in [KV86]. Obtaining bounds on the rate of convergence requires more work. For this, we need to identify the limiting Brownian motion and to introduce an additional Poisson equation.

The sequel of this paper is organized as follows. In section 2 we introduce the notation that we will be using throughout the paper and we present our main result, Theorem 2.1. In section 3 we prove various estimates on the invariant measure of (1.1) and the solution of the cell problem, and we also derive estimates on moments of the particle velocity. The proof of Theorem 2.1 is presented in section 4. Finally, section 5 is reserved for a few concluding remarks.

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## 2 Notation and Results

Consider the following Langevin equation in $\mathbb{R}^n$:

$$\tau\ddot{x}(t) = v(x(t)) - \dot{x}(t) + \sigma\dot{\beta}(t)\;, \tag{2.1}$$

with initial conditions $x(0) = x$, $\dot{x}(0) = (\sqrt{\tau})^{-1}y$. We assume throughout this paper that $v \in \mathcal{C}^\infty(\mathbb{T}^n)$. Introducing $y(t) = \sqrt{\tau}\dot{x}(t)$, we rewrite (2.1) as a first order stochastic differential equation:

$$\begin{aligned} dx(t) &= \frac{1}{\sqrt{\tau}}y(t)\,dt\;,\\ dy(t) &= \frac{1}{\sqrt{\tau}}v(x(t))\,dt - \frac{1}{\tau}y(t)\,dt + \frac{\sigma}{\sqrt{\tau}}d\beta(t)\;. \end{aligned} \tag{2.2}$$

We denote by $\mathcal{L}$ the generator of the process $\{x(t), y(t)\}$:

$$\mathcal{L} = \frac{1}{\sqrt{\tau}}\left(y\cdot\nabla_x + v(x)\cdot\nabla_y\right) + \frac{1}{\tau}\left(-y\cdot\nabla_y + \frac{\sigma^2}{2}\Delta_y\right). \tag{2.3}$$

By Theorem 3.1 below, the process $\{x(t), y(t)\}$ admits a unique, smooth invariant measure, denoted by $\mu(dx, dy)$.

Consider now the *cell problem*

$$-\mathcal{L}\Phi = \frac{1}{\sqrt{\tau}}y\;. \tag{2.4}$$

This equation has a unique, smooth solution in the appropriate function space by Theorem 3.3 , provided that $\int v(x)\,\mu(dx, dy) = 0$. We define the symmetric, positive $n\times n$ matrix $\mathcal{K}$ such that

$$\mathcal{K}^2 = \frac{\sigma^2}{\tau}\int \nabla_y\Phi\otimes\nabla_y\Phi\,d\mu\;. \tag{2.5}$$

The main result of this paper is that the particle position, under the standard diffusive rescaling, converges weakly to a Brownian motion with covariance $\mathcal{K}^2$. We furthermore give upper bounds on the rate of convergence in the following metric. Let $\mathrm{B}$ denote a separable Banach space and $\mathrm{B}^*$ be its dual space. Given two measures $\mu_1$ and $\mu_2$ on $\mathrm{B}$, we also denote by $\mathrm{C}(\mu_1, \mu_2)$ the set of all measures on $\mathrm{B}^2$ with marginals $\mu_1$ and $\mu_2$. With these notations, we define the following metric on the space of probability measures on $\mathrm{B}$ with finite $p$-moment:

$$|\!|\!|\mu_1 - \mu_2|\!|\!|_p^p = \sup_{\ell\in\mathrm{B}^*}\inf_{\mu_\ell\in\mathrm{C}(\mu_1,\mu_2)}\int_{\mathrm{B}^2}\frac{|\ell(x)-\ell(y)|^p}{\|\ell\|^p}\,\mu_\ell(dx, dy)\;. \tag{2.6}$$

This distance is close in spirit to the $p$-Wasserstein distance

$$|\!|\!|\mu_1 - \mu_2|\!|\!|_{p,W}^p = \inf_{\mu\in\mathrm{C}(\mu_1,\mu_2)}\int_{\mathrm{B}^2}\|x-y\|^p\,\mu(dx, dy)\;,$$

so we will refer to it as the weak $p$-Wasserstein distance. Note that the distance (2.6) gives a locally uniform bound on the distance between characteristic functions $\chi_\mu(\ell) = \int e^{i\ell(x)}\,\mu(dx)$:

$$|\chi_{\mu_1}(\ell) - \chi_{\mu_2}(\ell)| \le \|\ell\|\;|\!|\!|\mu_1 - \mu_2|\!|\!|_p\;.$$

In particular one has $\|\mu_1 - \mu_2\|_p = 0$ if and only if $\mu_1 = \mu_2$.

In order to simplify notations, we define the fast processes $y_t^\varepsilon = y(\varepsilon^{-2}t)$ and $x_t^\varepsilon = x(\varepsilon^{-2}t)$. We will also from now on use the notation $\mathrm{B} = \mathcal{C}([0,T],\mathbb{R}^n)$, for a value $T > 0$ that remains fixed throughout this paper. Now we are ready to state the homogenization theorem.

**Theorem 2.1** *Let $x(t)$ be the solution of (2.1), in which the velocity field $v \in \mathcal{C}^\infty(\mathbb{T}^n)$ satisfies $\int v(x)\,\mu(dx,dy) = 0$. For $T > 0$ fixed, denote by $\mu_\varepsilon$ the measure on $\mathrm{B}$ given by the law of the rescaled process $\varepsilon x_t^\varepsilon$ and by $\mu$ the law of a Brownian motion on $\mathbb{R}^n$ with covariance $\mathcal{K}^2$ as defined in (2.5). Then, for every $p \geq 1$ and $\alpha \in \big(0,\frac{1}{2}\big)$, there is a constant $C$ such that*

$$\|\mu_\varepsilon - \mu\|_p \leq C\varepsilon^\alpha \;, \tag{2.7}$$

*for all $\varepsilon \in (0,1)$. Furthermore, if one denotes by $\pi_k : \mathrm{B} \to \mathcal{C}([0,T],\mathbb{R})$ the projection given by $(\pi_k x)(t) = \langle k, x(t)\rangle$, one has the bound*

$$\|\pi_k^*\mu_\varepsilon - \pi_k^*\mu\|_{p,W} \leq C\varepsilon^\alpha \;, \tag{2.8}$$

*for every $k \in \mathbb{R}^n$ with $\|k\| \leq 1$.*

**Remark 2.2** The condition $\int v(x)\,\mu(dx,dy) = 0$ ensures that there is no ballistic motion involved. In the general case, one can write $\bar v = \int v(x)\,\mu(dx,dy)$ and define $\varepsilon x_t^\varepsilon = \varepsilon x(\varepsilon^{-2}t) - \varepsilon^{-1}\bar v t$. Then, Theorem 2.1 holds for $\varepsilon x_t^\varepsilon$.

**Remark 2.3** If $n = 1$, the bound (2.8) is much stronger than the bound (2.7). If $n > 1$ however, this bound does not imply any form of convergence $\mu_\varepsilon \Rightarrow \mu$. It is indeed possible to construct two Gaussian stochastic processes $x(t)$ and $y(t)$ with values in $\mathbb{R}^2$ such that the law of $x$ differs from the law of $y$ and such that, for every $k \in \mathbb{R}^2$, the law of $\langle k, x\rangle$ is identical to the law of $\langle k, y\rangle$. As an example, choose three i.i.d. Gaussian centered random variables $a_1, a_2, a_3$ and define

$$\begin{array}{cccc} x_1(t_1) = a_1 & x_2(t_1) = a_2 & x_1(t_2) = a_3 & x_2(t_2) = a_1 \\ y_1(t_1) = a_1 & y_2(t_1) = a_2 & y_1(t_2) = a_2 & y_2(t_2) = a_3 \;. \end{array}$$

It is an easy exercise to check that these two processes possess the required properties.

**Remark 2.4** Convergence in the weak $p$-Wasserstein distance alone does *not* imply weak convergence, as the space of probability measures on $\mathrm{B}$ is not complete under $\|\cdot\|_p$. This can be seen by taking $\mathrm{B} = \ell^2$ and choosing for $\mu_n$ the Gaussian measure with covariance

$$Q_n = \mathrm{diag}(1,\tfrac{1}{2},\ldots,\tfrac{1}{n},0,\ldots)\;.$$

It is straightforward to check that this forms a Cauchy sequence with respect to $\|\cdot\|_p$, but does not converge to any measure supported in $\ell^2$. (It does however converge weakly to a limiting measure in a weaker topology, and this is always the case.) In our case, it is easy to check that the sequence of measures $\mu_\varepsilon$ is tight, since the generalized Kolmogorov criteria [RY99, Thm 2.1] provides us with uniform bounds on the $\alpha$-Hölder constant (with $\alpha < \frac{1}{2}$) of the process $x_\varepsilon$. Tightness, together with convergence in the weak $p$-Wasserstein distance then implies weak convergence. Note also that even though convergence in the weak $p$-Wasserstein distance alone does not imply weak convergence, it does imply weak convergence of finite-dimensional distributions.

**Remark 2.5** The covariance, or *effective diffusivity*, $\mathcal{K}^2$ of the limiting Brownian motion depends on the $\sigma$ and $\tau$. It is shown in [PS03] that as $\tau$ tends to $0$ the covariance $\mathcal{K}^2$ converges to the one obtained from the homogenization of equation 1.2. We refer to [PS03] for further properties of the effective diffusivity, together with numerical experiments for various fields $v(x)$.

**Remark 2.6** For simplicity, we choose the molecular diffusion $\sigma$ to be a constant scalar. Taking for $\sigma$ a positive definite matrix would only require a slight change in our notations. We could even allow $\sigma$ to depend on $x$ in a smooth way, as long as it remains strictly positive definite for all $x \in \mathbb{T}^n$. The results from [EPRB99, EH00] then still apply and one can check that all the bounds obtained in section 3 still hold. Since the proof of Theorem 2.1 itself never uses the fact that $\sigma$ is constant, all of our result immediately carry over to this case.

**Remark 2.7** For simplicity, we assumed the initial condition $(x, y)$ to be deterministic. However, it is easy to check that all our arguments work for randomly distributed initial conditions provided that they are independent of the driving noise and that $\mathbf{E} \exp \delta \|y\|^2 < \infty$ for all $\delta \in (0, \sigma^{-2})$. In particular, one can take the initial condition to be distributed according to the invariant measure.

The proof of this theorem will be presented in section 4.

## 3 Preliminary Estimates

In this section we collect various estimates which are necessary for the proof of the homogenization theorem. In section 3.1 we study the structure of the invariant measure $\mu$ for (2.1). We show that it possesses a smooth density with respect to the Lebesgue measure and we derive sharp bounds for it. Further, we investigate the solvability of the Poisson equation

$$-\mathcal{L}f = h, \tag{3.1}$$

where $h$ is a smooth function of $x$ and $y$ which is centered with respect to $\mu$. We prove that equation (3.1) has a smooth solution which is unique in the class of functions which do not grow too fast at infinity.

In section 3.2 we derive estimates on exponential moments of the particle velocity. Roughly speaking, these estimates imply that the particle velocity grows very slowly with time.

### 3.1 Bounds on the invariant measure and on the solution of the Poisson equation

If $v = 0$, the invariant measure for (2.1) is given by $\mu = e^{-\frac{\|y\|^2}{\sigma^2}}\, dx\, dy$. This is "almost" true also in the case $v \neq 0$, as can be seen by the following result.

**Theorem 3.1** *Let $\mu$ be the invariant measure for (2.1) and denote by $\rho(x, y)$ its density with respect to the Lebesgue measure. Then, for every $\delta \in (0, 2\sigma^{-2})$ one can write*

$$\rho(x, y) = e^{-\frac{\delta}{2}\|y\|^2} g(x, y) , \qquad g \in \mathcal{S} , \tag{3.2}$$

*where $\mathcal{S}$ denotes the Schwartz space of smooth functions with fast decay.*

*Proof.* The proof follows the lines of [EPRB99, EH00]. Denote by $\phi_t$ the (random) flow generated by the solutions to (2.1) and by $\mathcal{P}_t$ the semigroup defined on finite measures by

$$(\mathcal{P}_t\mu)(A) = \mathbf{E}(\mu \circ \phi_t^{-1})(A) .$$

Since $\partial_t + \mathcal{L}$ is hypoelliptic, $\mathcal{P}_t$ maps every measure into a measure with a smooth density with respect to the Lebesgue measure. It can therefore be restricted to a positivity preserving contraction semigroup on Ł$^1(\mathbb{T}^n \times \mathbb{R}^n, dx\, dy)$. The generator $\tilde{\mathcal{L}}$ of $\mathcal{P}_t$ is given by the formal adjoint of $\mathcal{L}$ defined in (2.3).

We now define an operator $K$ on Ł$^2(\mathbb{T}^n \times \mathbb{R}^n, dx\, dy)$ by closing the operator defined on $\mathcal{C}_0^\infty$ by

$$K = -e^{\frac{\delta}{2}\|y\|^2}\tilde{\mathcal{L}}e^{-\frac{\delta}{2}\|y\|^2} . \tag{3.3}$$

The operator $K$ is then given by

$$\begin{aligned} K = &-\frac{\sigma^2}{2\tau}\Delta_y + \frac{\delta}{\tau}\Big(1 - \frac{\delta\sigma^2}{2}\Big)\|y\|^2 + \frac{1}{\tau}(\delta\sigma^2 - 1)\Big(y\cdot\nabla_y + \frac{n}{2}\Big) \\ &+ \frac{1}{\sqrt{\tau}}(y\cdot\nabla_x + v(x)\cdot\nabla_y) - \frac{n}{2\tau} . \end{aligned}$$

Note at this point that $\delta < 2\sigma^{-2}$ is required to make the coefficient of $\|y\|^2$ in this expression strictly positive. This can be written in Hörmander's "sum of squares" form as

$$K = \sum_{i=1}^{2n} X_i^* X_i + X_0 ,$$

with

$$\begin{aligned} X_i &= \frac{\sigma}{\sqrt{2\tau}}\partial_{y_i} && \text{if } i = 1\ldots n, \\ X_i &= \sqrt{\frac{\delta}{\tau}\Big(1 - \frac{\delta\sigma^2}{2}\Big)}\,y_{i-n} && \text{if } i = (n+1)\ldots 2n, \\ X_0 &= \frac{1}{\tau}(\delta\sigma^2 - 1)\Big(y\cdot\nabla_y + \frac{n}{2}\Big) + \frac{1}{\sqrt{\tau}}(y\cdot\nabla_x + v(x)\cdot\nabla_y) - \frac{n}{2\tau} . \end{aligned}$$

Since $v$ is $\mathcal{C}^\infty$ on the torus, it can be checked in a very straightforward way that the assumptions of [EH00, Thm. 5.5] are satisfied with $\Lambda^2 = 1 - \Delta_x - \Delta_y + \|y\|^2$. Combining this with [EH00, Lem. 5.6], we see that there exists $\alpha > 0$ such that, for every $\gamma > 0$, there exista a positive constant $C$ such that

$$\|\Lambda^{\alpha+\gamma} f\| \le C(\|\Lambda^\gamma K f\| + \|\Lambda^\gamma f\|) , \tag{3.4}$$

holds for every $f$ in the Schwartz space. Looking at (3.4) with $\gamma = 0$, we see that $K$ has compact resolvent. Since $e^{-\frac{\delta}{2}\|y\|^2}$ is an eigenfunction with eigenvalue $0$ for $K^*$, it follows that $K$ has also an eigenfunction with eigenvalue $0$, let us call it $g$. It follows from (3.4) and a simple approximation argument that $\|\Lambda^\gamma g\| < \infty$ for every $\gamma > 0$, and therefore $g$ belongs to the Schwartz space. Furthermore, an argument given for example in [EPRB99, Prop 3.6] shows that $g$ must be positive. Since one has furthermore

$$\tilde{\mathcal{L}}e^{-\frac{\delta}{2}\|y\|^2} g = 0 ,$$

the function $\rho$ given by (3.2) is the density of the invariant measure of (2.1). This concludes the proof of Theorem 3.1. $\square$

Before we give bounds on (2.4), we show the following little lemma.

**Lemma 3.2** *Let $\delta \in (0, 2\sigma^{-2})$ and let $K$ be as in (3.3). Then, the kernel of $K$ is one-dimensional.*

*Proof.* Let $\tilde{g} \in \ker K$. Then, by the same arguments as above, $e^{-\frac{\delta}{2}\|y\|^2}\tilde{g}$ is the density of an invariant signed measure for $\mathcal{P}_t$. The ergodicity of $\mathcal{P}_t$ immediately implies $\tilde{g} \propto g$. □

Now we are ready to prove estimates on the solution of the Poisson equation (3.1).

**Theorem 3.3** *Let $h \in \mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ with $D^\alpha_{x,y} h \in L^2(\mathbb{T}^n \times \mathbb{R}^n; e^{-\varepsilon\|y\|^2}dxdy)$ for every multiindex $\alpha$ and every $\varepsilon > 0$. Assume further that $\int h(x, y)\, \mu(dx\, dy) = 0$, where $\mu$ is the invariant measure for (2.1). Then, there exists a function $f$ such that (3.1) holds. Moreover, for every $\delta > 0$, the function $f$ satisfies*

$$f(x, y) = e^{\frac{\delta}{2}\|y\|^2}\tilde{f}(x, y)\;, \qquad \tilde{f} \in \mathcal{S}\;. \tag{3.5}$$

*Furthermore, for every $\delta \in (0, 2\sigma^{-2})$, $f$ is unique (up to an additive constant) in $L^2(\mathbb{T}^n \times \mathbb{R}^n, e^{-\delta\|y\|^2}dxdy)$.*

*Proof.* By hypoellipticity, if there exists a distribution $f$ such that (2.4) holds, then $f$ is actually a $\mathcal{C}^\infty$ function.

We start with the proof of existence. Fix $\delta \in (0, 2\sigma^{-2})$, consider the operator $K^*$ which is the adjoint of the operator $K$ defined in (3.3), and define the function

$$u(x, y) = h(x, y)\, e^{-\frac{\delta}{2}\|y\|^2}\;.$$

It is clear that if there exists $\tilde{f}$ such that $K^*\tilde{f} = u$, then $f = e^{\frac{\delta}{2}\|y\|^2}\tilde{f}$ is a solution to (3.1). Consider the operator $K^*K$. By the considerations in the proof of Theorem 3.1, $K^*K$ has compact resolvent. Furthermore, the kernel of $K^*K$ is equal to the kernel of $K$, which in turn by Lemma 3.2 is equal to the span of $g$. Define $\mathcal{H} = \langle g\rangle^\perp$ and define $M$ to be the restriction of $K^*K$ to $\mathcal{H}$. Since $K^*K$ has compact resolvent, it has a spectral gap and so $M$ is invertible. Furthermore, since $\mathcal{L}y = \tau^{-1/2}v(x) - \tau^{-1}y$, one checks easily that $f \in \mathcal{H}$, therefore $\tilde{f} = KM^{-1}u$ solves $K^*\tilde{f} = u$ and thus leads to a solution to (3.1).

Since $K^*$ satisfies a similar bound to (3.4) and since $\|\Lambda^\gamma u\| < \infty$ for every $\gamma > 0$, the bound (3.5) follows as in Theorem 3.1. The uniqueness of $u$ in the class of functions under consideration follows immediately from Lemma 3.2. □

**Remark 3.4** Note that the solution $f$ of (3.1) is probably not unique if we allow for functions that grow faster than $e^{\sigma^{-2}\|y\|^2}$.

**Remark 3.5** The identity $y\tilde{\mathcal{L}}\rho = 0$, where $\tilde{\mathcal{L}}$ is the formal adjoint of $\mathcal{L}$, immediately yields that $\int y\, \mu(dx, dy) = \sqrt{\tau}\int v(x)\, \mu(dx, dy)$. In particular, the assumption that the drift is centered implies that $y$ is also centered. Moreover, $y$ clearly satisifies the smoothness and fast decay assumptions of Theorem 3.3. Hence, the theorem applies to each component of equation (2.4) and we can conclude that there exists a unique smooth vector valued function $\Phi$ which solves the cell problem and whose components satisfiy estimate (3.5).

### 3.2 Estimates on the particle velocity

One has the following bound

**Lemma 3.6** *There exists a constant $\gamma > 0$ such that*

$$\mathbf{E}\exp\Big(\frac{1}{2}\|\sigma^{-1}y(t)\|^2\Big) \le \exp\Big(\frac{1}{2}\|\sigma^{-1}y(0)\|^2 + \gamma t\Big) ,$$
$$\mathbf{E}\exp\Big(\frac{1}{8\tau}\int_0^t \|\sigma^{-1}y(s)\|^2\,ds\Big) \le \exp\Big(\frac{1}{4}\|\sigma^{-1}y(0)\|^2 + \frac{\gamma}{2}t\Big) .$$

*holds for any initial condition $y(0)$ and every $t > 0$.*

*Proof.* Itôs formula yields immediately the existence of a constant $\gamma$ such that

$$\begin{aligned}\frac{1}{2}\|\sigma^{-1}y(t)\|^2 \le \frac{1}{2}\|\sigma^{-1}y(0)\|^2 + \gamma t \\ - \frac{1}{2\tau}\int_0^t \|\sigma^{-1}y(s)\|^2\,ds + \frac{1}{\sqrt{\tau}}\int_0^t \langle\sigma^{-1}y(s), d\beta(s)\rangle .\end{aligned}$$

The first bound follows by exponentiating both sides and taking expectations. The second bound follows in a similar way after dividing both sides by 2. □

This yields the following:

**Theorem 3.7** *Let $\psi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\sup_{x\in\mathbb{T}^n, y\in\mathbb{R}^n} \Big|\psi(x,y)\exp\Big(-\frac{1}{4}\|\sigma^{-1}y\|^2\Big)\Big| < \infty .$$

*Then, there exist constants $C, \delta > 0$ such that*

$$\mathbf{E}(\psi(x(t),y(t))) - \int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,y)\,\mu(dx,dy) \le C\exp(\|\sigma^{-1}y(0)\|^2 - \delta t) . \tag{3.6}$$

*Proof.* From the smoothing properties of the transition semigroup associated to (2.2), combined with its controllability and the fact that $\|y\|^2$ is a Lyapunov function, one gets the existence of constants $C$ and $\delta'$ such that

$$\|\mathcal{P}_t(x,y;\cdot\,) - \mu\|_{\mathrm{TV}} \le C(1 + \|y\|^2)e^{-\delta' t} .$$

(See *e.g.* [MT93] for further details.). Here $\|\nu - \mu\|_{\mathrm{TV}}$ denotes the total variation distance between the measures $\mu$ and $\nu$. Cauchy-Schwarz furthermore yields the generic inequality

$$\Big|\int f\,d\mu - \int f\,d\nu\Big| \le \sqrt{\|\mu-\nu\|_{\mathrm{TV}}\int f^2\,(d\mu + d\nu)} \tag{3.7}$$

The bound (3.6) immediately follows by combining Lemma 3.6 with (3.7). □

We also have a much stronger bound on the supremum in time of the solution:

**Lemma 3.8** *For every $\kappa > 0$ and every $T > 0$, there exist constants $\delta, C > 0$ such that*

$$\mathbf{E}\sup_{t\in[0,T\varepsilon^{-2}]} \exp(\delta\|y(s)\|^2) \le C\varepsilon^{-\kappa}e^{\delta\|y(0)\|^2} ,$$

*holds for every $\varepsilon \in [0,1]$.*

*Proof.* Let $\tilde{y}$ be the Ornstein-Uhlenbeck process defined by

$$\tilde{y}(t) = \frac{1}{\sqrt{\tau}} \int_0^t e^{-\frac{t-s}{\tau}} \sigma \, d\beta(s) \; .$$

Then (see *e.g.* [Adl90]), there exists constants $c_1$ and $c_2$ such that

$$\mathbf{P}\Big( \sup_{t\in[s,s+T]} \|\tilde{y}(t)\| > \lambda \Big) \le c_1 e^{-c_2 \lambda^2} \; ,$$

for every $s > 0$. This immediately yields

$$\mathbf{P}\Big( \sup_{t\in[0,T\varepsilon^{-2}]} \|\tilde{y}(t)\| > \lambda \Big) \le c_1 \varepsilon^{-2} e^{-c_2 \lambda^2} \; ,$$

which in turn implies that there exist constants $c_3$ and $c_4$ such that

$$\mathbf{E}\Big( \sup_{t\in[0,T\varepsilon^{-2}]} \exp(c_3 \|\tilde{y}(t)\|^2) \Big) \le c_4 \varepsilon^{-2} \; .$$

The claim follows immediately by choosing $\delta = (c_3\kappa)/2$ and by noticing that there exists a constant $c_4$ such that $\|y(s)\| \le \|\tilde{y}(s)\| + \|y(0)\| + c_4$ for all $s > 0$ almost surely. □

## 4 Proof of Theorem 2.1

In this section we prove Theorem 2.1.

*Proof.* By Theorem 3.3 we have $\Phi(y,z) \in \mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n, \mathbb{R}^n)$, so we can apply the Itô formula to the function $\Phi(y_t^\varepsilon, x_t^\varepsilon)$ to obtain:

$$\begin{aligned} \Phi(y_t^\varepsilon, x_t^\varepsilon) - \Phi(y,x) &= \frac{1}{\varepsilon^2} \int_0^t \mathcal{L}\Phi(y_s^\varepsilon, x_s^\varepsilon) \, ds + \frac{1}{\varepsilon} \frac{\sigma}{\sqrt{\tau}} \int_0^t \nabla_y \Phi(y_s^\varepsilon, x_s^\varepsilon) \, d\beta^\varepsilon(s) \\ &= -\frac{1}{\varepsilon^2} \frac{1}{\sqrt{\tau}} \int_0^t y_s^\varepsilon \, ds + \frac{1}{\varepsilon} \frac{\sigma}{\sqrt{\tau}} \int_0^t \nabla_y \Phi(y_s^\varepsilon, x_s^\varepsilon) \, d\beta^\varepsilon(s) \; , \end{aligned}$$

where we defined $\beta^\varepsilon(t) = \varepsilon\beta(\varepsilon^{-2}t)$ and we used (2.4) to get the second line. We also interpret $\nabla_y \Phi$ as a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$. Thus we have:

$$\begin{aligned} \varepsilon x_t^\varepsilon &= \varepsilon x + \frac{1}{\varepsilon} \frac{1}{\sqrt{\tau}} \int_0^t y_s^\varepsilon \, ds \\ &= \varepsilon x - \varepsilon(\Phi(y_t^\varepsilon, x_t^\varepsilon) - \Phi(y,x)) + \frac{\sigma}{\sqrt{\tau}} \int_0^t \nabla_y \Phi(y_s^\varepsilon, x_s^\varepsilon) \, d\beta^\varepsilon(s) \\ &=: \varepsilon x + \varepsilon I_1^\varepsilon(t) + M^\varepsilon(t) \; . \end{aligned} \tag{4.1}$$

It follows from (3.5) and Lemma 3.8 that, for every $p > 0$ there exists a constant $C$ such that

$$\mathbf{E} \sup_{t\in[0,T]} |I_1^\varepsilon(t)|^p \le C\varepsilon^{-\frac{p}{2}} \; .$$

It is therefore sufficient to show that (2.7) and (2.8) hold with $\mu_\varepsilon$ replaced by the law of the martingale term $M^\varepsilon$. We first show that (2.7) holds. This is equivalent to showing that, for every $\ell \in \mathrm{B}^*$ one can construct a random variable $B_\ell$ such that

$$\mathbf{E}|B_\ell - \ell(M^\varepsilon)|^p \le C\varepsilon^{\alpha p} \; , \tag{4.2}$$

holds uniformly over $\|\ell\| \le 1$, and such the law of $B_\ell$ is given by $\ell^*\mu$. We therefore fix $\ell \in \mathrm{B}^*$ with $\|\ell\| \le 1$, which we interpret as a $\mathbb{R}^n$-valued measure with total mass (*i.e.* the sum of the masses of each of its components) smaller than $1$. We also use the notation $\ell_t = \ell([t,T])$.

Integrating by parts, we can write

$$\ell(M^\varepsilon) = \int_0^T \langle M^\varepsilon(t), \ell(dt)\rangle = \frac{\sigma}{\sqrt{\tau}} \int_0^T \langle \ell(t), \nabla_y \Phi(y_t^\varepsilon, x_t^\varepsilon)\, d\beta^\varepsilon(t)\rangle \;.$$

We now define on the interval $[0,T]$ the $\mathbb{R}$-valued martingale $M_\ell^\varepsilon$ by

$$M_\ell^\varepsilon(t) = \frac{\sigma}{\sqrt{\tau}} \int_0^t \langle \ell(s), \nabla_y \Phi(y_s^\varepsilon, x_s^\varepsilon)\, d\beta^\varepsilon(s)\rangle \;.$$

According to the Dambis–Dubins–Schwartz theorem [RY99, Thm 1.6] there exists a Brownian motion $B$ such that $M_\ell^\varepsilon(t)$ can be written as

$$M_\ell^\varepsilon(t) = B(\langle M_\ell^\varepsilon, M_\ell^\varepsilon\rangle_t) = B\Big(\frac{\sigma^2}{\tau}\int_0^t \Big\langle \ell(s), (\nabla_y\Phi \otimes \nabla_y \Phi)(y_s^\varepsilon, x_s^\varepsilon)\, \ell(s)\Big\rangle\, ds\Big) \;.$$

On the other hand, the measure $\ell^*\mu$ is a centered Gaussian measure with variance $\int_0^T \langle \ell(s), \mathcal{K}^2 \ell(s)\rangle\, ds$, so we can choose $B_\ell$ to be given by

$$B_\ell = B_\ell^T \;,\qquad B_\ell^t = B\Big(\int_0^t \langle \ell(s), \mathcal{K}^2\ell(s)\rangle\, ds\Big) \;.$$

We will actually show a stronger bound than (4.2), namely we will show that

$$J_\varepsilon^p := \mathbf{E} \sup_{t\in[0,T]} |B_\ell^t - M_\ell^\varepsilon(t)|^p \le C\varepsilon^{\alpha p} \;. \tag{4.3}$$

We use the Hölder continuity of the Brownian motion $B$, together with the Cauchy–Schwarz inequality to derive the estimate

$$\begin{aligned}
J_\varepsilon^p &\le \mathbf{E}\Big(\mathrm{H\ddot{o}l}_\alpha^p(B) \sup_{0\le t\le T}\Big|\int_0^t \Big\langle \ell(s), \Big(\frac{\sigma^2}{\tau}(\nabla_y\Phi\otimes\nabla_y\Phi)(y_s^\varepsilon, x_s^\varepsilon) - \mathcal{K}^2\Big)\,\ell(s)\Big\rangle\, ds\Big|^{\alpha p}\Big) \\
&\le (\mathbf{E}\,\mathrm{H\ddot{o}l}_\alpha^{2p}(B))^{\frac12}\Big(\mathbf{E}\sup_{0\le t\le T}\Big|\int_0^t \langle \ell(s), H(y_s^\varepsilon, x_s^\varepsilon)\,\ell(s)\rangle\, ds\Big|^{2\alpha p}\Big)^{\frac12} \\
&\le C\Big(\mathbf{E}\sup_{0\le t\le T}\Big|\int_0^t \langle \ell(s), H(y_s^\varepsilon, x_s^\varepsilon)\,\ell(s)\rangle\, ds\Big|^{2\alpha p}\Big)^{\frac12} \;,
\end{aligned} \tag{4.4}$$

where we introduced the $n\times n$-matrix valued function

$$H(x,y) = \frac{\sigma^2}{\tau}(\nabla_y\Phi\otimes\nabla_y\Phi)(y,x) - \mathcal{K}^2 \;.$$

In deriving the above estimate, we have used the fact that for $\alpha < \frac12$, the $\alpha$-Hölder constant of a Brownian motion is uniformly bounded on every bounded interval [RY99, Thm 2.1].

Note now that since $\ell(t)$ is of bounded variation, $\ell(t)\otimes\ell(t)$ is also of bounded variation, so there exists a $n\times n$-matrix valued measure $\tilde\ell$ on $[0,T]$ such that $\ell(t)\otimes\ell(t) = \tilde\ell([t,T])$. Therefore, we can integrate by parts in (4.4) to obtain

$$J_\varepsilon^p \le C\Big(\mathbf{E}\sup_{0\le t\le T}\Big|\mathrm{Tr}\int_0^t\int_0^s H(y_r^\varepsilon, x_r^\varepsilon)\, dr\, \tilde\ell(ds)\Big|^{2\alpha p}\Big)^{\frac12}$$

$$\leq C\Big(\mathbf{E}\sup_{0\leq t\leq T}\Big\|\int_0^t H(y_s^\varepsilon,x_s^\varepsilon)\,ds\Big\|^{2\alpha p}\Big)^{\frac{1}{2}}$$

Consider now the Poisson equation

$$-\mathcal{L}F = H\;. \tag{4.5}$$

By the definition of $\mathcal{K}^2$, we have $\int H(x,y)\,\mu(dx,dy) = 0$ (for each component), and we furthermore have $\exp(-\delta\|y\|^2)H \in \mathcal{S}$ for every $\delta > 0$. Therefore, using the same reasoning as in the proof of Theorem 3.3, equation (4.5) has a unique smooth solution satisfying

$$F(x,y) = e^{\frac{\delta}{2}\|y\|^2}\tilde{\mathcal{F}}(x,y)\;,\qquad \tilde{\mathcal{F}}\in\mathcal{S} \tag{4.6}$$

for every $\delta > 0$. We can apply Itô formula to deduce as before that

$$\int_0^t H(y_s^\varepsilon,x_s^\varepsilon)\,ds = -\varepsilon^2(F(y_t^\varepsilon,x_t^\varepsilon)-F(y,x)) + \frac{\varepsilon}{\sqrt{\tau}}\int_0^t \nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,\sigma d\beta(s)\;.$$

Therefore:

$$|J_\varepsilon^p|^2 \leq \varepsilon^{4\alpha p}\mathbf{E}\sup_{t\in[0,T]}\|F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p} + C\varepsilon^{2\alpha p}\mathbf{E}\sup_{t\in[0,T]}\Big\|\int_0^t \nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,d\beta(s)\Big\|^{2\alpha p}\;.$$

Combining Lemma 3.8 with (4.6), the first term can be bounded by

$$\varepsilon^{4\alpha p}\mathbf{E}\sup_{t\in[0,T]}\|F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p} \leq C\varepsilon^{-2\alpha p}\;.$$

In order to control the second term, we use the Burkholder–Davis–Gundy inequality followed by Hölder's inequality, assuming that $p > \frac{1}{\alpha}$:

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,T]}\Big\|\int_0^t \nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,d\beta(s)\Big\|^{2\alpha p} &\leq C\mathbf{E}\Big(\int_0^T \|\nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\|^2\,ds\Big)^{\alpha p}\\ &\leq CT^{\alpha p-1}\sup_{t\in[0,T]}\mathbf{E}\|\nabla_y F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p}\;.\end{aligned}$$

This is bounded independently of $\varepsilon$ by (4.6) and Lemma 3.6, and so $J_\varepsilon^p \leq C\varepsilon^{\alpha p}$, for $p > \frac{1}{\alpha}$. When $p < \frac{1}{\alpha}$, one can bound $J_\varepsilon^p$ using the higher order moments. This completes the proof of bound (4.2) and thus of the first part of Theorem 2.1.

The proof of the second part of Theorem 2.1 is obtained in a straightforward way as a particular case of (4.3) if one makes the choice $\ell = k\delta_T$. □

## 5 Conclusions

The problem of homogenization for periodic hypoelliptic diffusions was studied in this paper. It was proved that the rescaled particle position converges to a Brownian motion with a covariance matrix which can be computed in terms of the solution of the Poisson equation (2.4). Further, an upper bound on the convergence rate in a suitable norm was obtained. Our analysis is purely probabilistic and this enables us to obtain more detailed information than what one could obtain from studying the problem at the level of the Kolmogorov equation.

A very interesting question is whether a homogenization theorem of the form 2.1 holds for random drifts $v(x,t)$ and, if yes, under what conditions on $v(x,t)$. From a mathematical point of view, it would be interesting to know whether it is possible to achieve convergence in the $p$-Wasserstein distance for $n > 1$. We plan to come back to these issues in a future publication.

## References

[Adl90] R. J. ADLER. *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, vol. 12 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1990.

[Bat85] R. BATTACHARYA. A central limit theorem for diffusions with periodic coefficients. *The Annals of Probability* **13**, (1985), 385–396.

[BBJR95] C. J. K. BATTY, O. BRATTELI, P. E. T. JØRGENSEN, and D. W. ROBINSON. Asymptotics of periodic subelliptic operators. *J. Geom. Anal.* **5**, no. 4, (1995), 427–443.

[BLP78] A. BENSOUSSAN, J. LIONS, and G. PAPANICOLAOU. *Asymptotic analysis of periodic structures*. North-Holland, Amsterdam, 1978.

[CX97] R. CARMONA and L. XU. Homogenization theory for time-dependent two-dimensional incompressible gaussian flows. *The Annals of Applied Probability* **7**, no. 1, (1997), 265–279.

[EH00] J.-P. ECKMANN and M. HAIRER. Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators. *Comm. Math. Phys.* **212**, no. 1, (2000), 105–164.

[EPRB99] J.-P. ECKMANN, C.-A. PILLET, and L. REY-BELLET. Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. *Comm. Math. Phys.* **201**, no. 3, (1999), 657–697.

[Fre01] M. I. FREIDLIN. On stable oscillations and equilibriums induced by small noise. *J. Statist. Phys.* **103**, no. 1-2, (2001), 283–300.

[FW98] M. FREIDLIN and M. WEBER. Random perturbations of nonlinear oscillators. *Ann. Probab.* **26**, no. 3, (1998), 925–967.

[FW99] M. FREIDLIN and M. WEBER. A remark on random perturbations of the nonlinear pendulum. *Ann. Appl. Probab.* **9**, no. 3, (1999), 611–628.

[FW01] M. FREIDLIN and M. WEBER. On random perturbations of Hamiltonian systems with many degrees of freedom. *Stochastic Process. Appl.* **94**, no. 2, (2001), 199–239.

[KO01] T. KOMOROWSKI and S. OLLA. On homogenization of time-dependent random flows. *Probab. Theory Related Fields* **121**, no. 1, (2001), 98–116.

[KV86] C. KIPNIS and S. R. S. VARADHAN. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104**, no. 1, (1986), 1–19.

[LOY98] C. LANDIM, S. OLLA, and H. T. YAU. Convection-diffusion equation with space-time ergodic random flow. *Probab. Theory Related Fields* **112**, no. 2, (1998), 203–220.

[MT93] S. P. MEYN and R. L. TWEEDIE. *Markov chains and stochastic stability*. Communications and Control Engineering Series. Springer-Verlag London Ltd., London, 1993.

[Nel67] E. NELSON. *Dynamical theories of Brownian motion*. Princeton University Press, Princeton, N.J., 1967.

[Nor94] J. R. NORRIS. Heat kernel bounds and homogenization of elliptic operators. *Bull. London Math. Soc.* **26**, no. 1, (1994), 75–87.

[Nor97] J. R. NORRIS. Long-time behaviour of heat flow: global estimates and exact asymptotics. *Arch. Rational Mech. Anal.* **140**, no. 2, (1997), 161–195.

[Par99] E. PARDOUX. Homogenization of linear and semilinear second order parabolic pdes with periodic coefficients: A probabilistic approach. *Journal of Functional Analysis* **167**, (1999), 498–520.

[PS03] G. A. PAVLIOTIS and A. STUART. Periodic homogenization for inertial particles. *Preprint* .

[PV85] G. PAPANICOLAOU and S. R. S. VARADHAN. Ornstein-Uhlenbeck process in a random potential. *Comm. Pure Appl. Math.* **38**, no. 6, (1985), 819–834.

[Rod89] H. RODENHAUSEN. Einstein's relation between diffusion constant and mobility for a diffusion model. *J. Statist. Phys.* **55**, no. 5-6, (1989), 1065–1088.

[RY99] D. REVUZ and M. YOR. *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third ed., 1999.